Computational Cognitive Science

Lecture 8: Model comparison and selection 2

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Readings

- Chapter 10.6 of F&L
- Chapter 11 of F&L

Optional:

- "Bayesian hypothesis testing for psychologists: A tutorial on the Savage–Dickey method" (2010) by Wagenmakers et al. (Link)

Model comparison

- We discussed some methods for comparing models using likelihoods under MLEs, and predictive accuracy

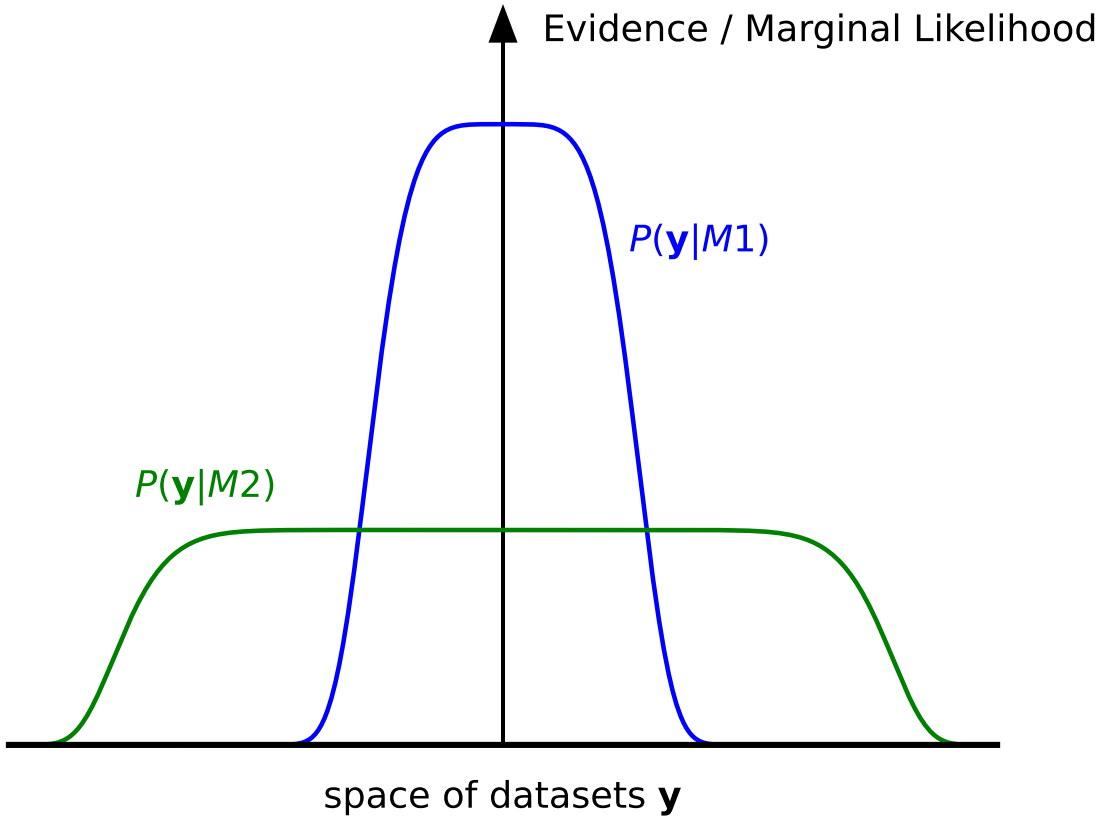
Bayesian model comparison

Last time we discussed approaches to model comparison that don't involve marginal likelihoods.

Today we'll talk about using and approximating marginal likelihoods:

$$p(\mathbf{y}|\mathcal{M}) = \int_{\boldsymbol{\theta}} p(\mathbf{y}|\boldsymbol{\theta}, \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M}) d\boldsymbol{\theta}$$

Bayesian model comparison



- More complex models spread their probability across more possible outcomes.
- Data $\mathbf{y}$ that both models can predict, tends to have higher marginal likelihood (evidence) under the simpler model.

Bayesian model comparison

Even if we have marginal likelihoods under several models, we can't compute $P(\mathcal{M}|\mathbf{y})$ ($P(\mathcal{M}|\mathbf{y}) = \frac{P(\mathbf{y}|\mathcal{M})P(\mathcal{M})}{p(\mathbf{y})}$):

We would need to sum over all possible models and data distributions for all of them. Instead:

- Given any two models $\mathcal{M}_1$ and $\mathcal{M}_1$, we can compute the ratio of their posterior probabilities:

$$\frac{P(\mathcal{M}_1|\mathbf{y})}{P(\mathcal{M}_2|\mathbf{y})} = \frac{P(\mathbf{y}|\mathcal{M}_1)P(\mathcal{M}_1)}{P(\mathbf{y}|\mathcal{M}_2)P(\mathcal{M}_2)}$$

The intractable/unknown normalization terms cancel out, and we get the *relative* probabilities of our models.

Bayesian model comparison

There is unlikely to be consensus about $P(\mathcal{M})$; in practice people use *Bayes factors* (BF):

$$\underbrace{\frac{P(\mathcal{M}_1|\mathbf{y})}{P(\mathcal{M}_2|\mathbf{y})}}_{\text{posterior odds}} = \underbrace{\frac{P(\mathbf{y}|\mathcal{M}_1)}{P(\mathbf{y}|\mathcal{M}_2)}}_{\text{BF}} \cdot \underbrace{\frac{P(\mathcal{M}_1)}{P(\mathcal{M}_2)}}_{\text{prior odds}}$$

- BF is the posterior ratio given equal prior probabilities
- How strongly you'd have to prefer a model *a priori* in order to (still) favor it *a posteriori*

Lots of opinions about what constitutes a "convincing" Bayes factor

Estimating marginal likelihoods

If we have no closed-form solution for a marginal likelihood, what can we do?

We have several options, including:

1. Numerical integration
2. Importance sampling
3. ~~Harmonic mean estimation~~
4. Transdimensional MCMC
5. Savage-Dickey density ratio
6. BIC

Numerical integration

Use a general-purpose algorithm to integrate a function within a hypercube.

- Easy!
- Requires bounds on the high-density parts of the space
- Intractable in high-dimensional spaces
- Risks missing narrow peaks

Numerical integration: Example

We can use standard probability densities to test our methods, since we know their integral (over **data**, not parameters) is 1.

Recall that the normal density function is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\mu-x)^2}{2\sigma^2}}$.

Let's approximate the integral of $e^{-\frac{(\mu-x)^2}{2\sigma^2}}$, which is the reciprocal of the normalizing constant $\frac{1}{\sqrt{2\pi\sigma^2}}$.

Numerical integration: Example

```
library(cubature)
sd=10;mu=.5
unnGauss <- function(x) {exp(-(mu-x)^2/(2*sd^2))}
adaptIntegrate(unnGauss,c(-1E3),c(1E3))
```

Result:

```
$integral
[1] 25.06628
```

```
> sqrt(2*pi*sd^2)
[1] 25.06628
```

(Also see F&L listing 11.1)

Importance sampling

Suppose we have:

- $p(\boldsymbol{\theta}|\mathcal{M})$ (prior over parameters)
- $p(\mathbf{y}|\boldsymbol{\theta}, \mathcal{M})$ (likelihood)

and we want the marginal likelihood:

- $p(\mathbf{y}|\mathcal{M}) = \int_{\boldsymbol{\theta}} p(\mathbf{y}|\boldsymbol{\theta}, \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M}) d\boldsymbol{\theta}$

as well as expected values for some psychologically interpretable parameters:

- $E[\boldsymbol{\theta}|\mathbf{y}, \mathcal{M}]$

Importance sampling

We can estimate these quantities using *importance sampling*:

(omitting $\mathcal{M}$)

1. Draw J samples from a **normalized** *proposal distribution* $\boldsymbol{\theta}^{(j)} \sim g(\boldsymbol{\theta})$
2. Weight each sample: $w_j = \frac{p(\mathbf{y}|\boldsymbol{\theta}^{(j)})p(\boldsymbol{\theta}^{(j)})}{g(\boldsymbol{\theta}^{(j)})}$
3. The expectation of $\boldsymbol{\theta}$ is approximately $\frac{\sum_j w_j \boldsymbol{\theta}^{(j)}}{\sum_j w_j}$
4. The marginal likelihood is approximately $\frac{1}{J}\sum_j w_j$

If the variance of the weights is low, these are *probably* trustworthy estimates.

Simple Monte Carlo integration

A special case of importance sampling:

1. Sample from the prior (usually easy)
2. Weight by likelihood: $w_j = \frac{p(\mathbf{y}|\boldsymbol{\theta}^{(j)})p(\boldsymbol{\theta}^{(j)})}{p(\boldsymbol{\theta}^{(j)})} = p(\mathbf{y}|\boldsymbol{\theta}^{(j)})$

Easy and acceptable if you think the samples will cover high-density areas of the posterior.

Better: Find a (normalized) proposal function that resembles your posterior.

Importance sampling

```
impSamp <- function(targD,ef) {
  nSamps = 40000 # The more the better
  # Using a student's t distribution, df=1
  proposals <- rt(nSamps,1)
  pDens <- dt(proposals,1)
  unnP <- targD(proposals)
  w <- unnP/pDens
  print(paste("Expected value of target function:",
    sprintf("%2.3f",sum(w*ef(proposals))/sum(w))))
  print(paste("Average importance weight:",
    sprintf("%2.3f",sum(w)/nSamps)))
}
```

Gaussian example

```r
sd=.05;mu=5;
unnGauss <- function(x) {exp(-(mu-x)^2/(2*sd^2))}
# Real
print(sprintf("Real: %2.3f",sqrt(2*pi*sd^2)))
# Numerical
adaptIntegrate(unnGauss,c(-1E3),c(1E3))
# Importance
impSamp(unnGauss,function(x) x)
```

Gaussian example

```
sd=.05;mu=5

Real normalization constant Z: 0.125

Cubature estimate of Z: 0 (oops)

Importance sampling:
  Estimated mean: 5.000
  Avg. importance weight (Z): 0.132 (close)
```

Importance sampling

In the Gaussian example, we estimated the integral of the unnormalized Gaussian, by integrating over x.

$$\int e^{-\frac{(\mu-x)^2}{2\sigma^2}} dx$$

If we were interested in the marginal likelihood, we would propose μ and σ (i.e., the parameters $\boldsymbol{\theta} = [\mu, \sigma]$) rather than x. For this we would need proposal distributions and priors for both parameters.

$$p(x) = \int \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\mu-x)^2}{2\sigma^2}} p(\mu, \sigma) d\mu d\sigma$$

Importance sampling

- General-purpose Monte Carlo method for approximating parameter distributions
- Can exploit knowledge about high-density regions of posterior
- Can compute expectations of functions of params
- Requires good proposals
  - Increasingly so as dimensionality goes up
- In these cases, additional tricks may be necessary, e.g.,
  - **annealed** importance sampling (link)
  - Inference trees (link)

Harmonic mean estimation

See "The Harmonic Mean of the Likelihood: Worst Monte Carlo Method Ever" by Radford Neal. (link)

Excerpts:

- "abysmal performance in most real problem[s]"
- "the total unsuitability of the harmonic mean estimator should have been apparent within an hour of its discovery"

Don't use it.

Transdimensional MCMC

- Use Markov Chain Monte Carlo, combining multiple models into a single overarching one
- Nice in principle
- Often difficult / fiddly in practice
- Out of scope for this course

Savage-Dickey density ratio

- Efficiently compare nested probabilistic models via Bayes Factor
- Effectively a better and Bayesian alternative to the likelihood-ratio test
  - See recommended reading to learn more

BIC

$$BIC = -2 \cdot LL + K \log(N)$$

where N is the number of data points and K is the number of parameters.

- Motivated by model comparison per se, not prediction
- Approximates the log marginal likelihood under a set of assumptions
- Can be understood as a "minimum description length" approach
- Like AIC, a model-comparison method that boils down to MLE likelihoods and counting parameters
- Like AIC, rests on assumptions; guarantees are asymptotic
- Easy!
- Safer than AIC if arguing for a more complex model

Identifiability

"Models are unidentifiable when there is no unique mapping between any possible data pattern and a corresponding set of parameter estimates"

Strict identifiability is a high bar. More pragmatically:

- We should be able to identify parameters *we care about*
- There may be some patterns of data that don't lead to unique parameter estimates

Identifiability

If likelihood doesn't depend strongly on parameters:

- Effectively a simpler model
- Parameters may not be identifiable in practice

Identifiability

Weak relationships between parameters and data are not all bad:

- Less likely to have unwanted flexibility
- Sometime we want parameters *not* to matter
  - "Nuisance parameters"
  - Ideally we integrate them out and forget about them

Identifiability

However:

- If parameters have important psychological interpretations, they should be identifiable
- Might be a sign that the experiment isn't adequate

Identifiability

1. Sometimes non-identifiability is inherent in a model; function from parameters to data isn't invertible
   - E.g., using mean response times to infer parameters of a Weibull distribution
2. Sometimes identification is impossible in practice, because data are too sparse or noisy

Identifiability

If identifiability is important, we can perform an identifiability "sanity check" before collecting data.

This is sometimes possible to do mathematically, e.g., Jacobian rank (See F&L 10.6.1), but **simulation**-based approaches are often easier and more useful.

Fake data simulation

1. Define your model(s) and decide on how you will estimate parameters and compare models
2. Choose some theoretically interesting and plausible hypothetical models and/or parameters
3. Simulate data for your experiment based on each of those hypotheticals
   - Inspect your simulated data. Do they look implausible? If so, revisit steps 1-2
4. Compare models/fit parameters given your simulated data
   - Do you come to the right conclusion?
   - If not, you need to fix your models, methods, and/or experiments

Can also serve as a power analysis.

Summary

- Many methods for approximating the marginal likelihood
- Easy cases:
  - low-dimensional models (numerical integration, importance sampling)
  - nested models (Savage-Dickey)
  - conjugate priors (didn't discuss)
- Hard case: High-dimensional, non-conjugate, non-nested
  - Well-tuned importance sampling
  - Annealed importance sampling (out of scope)
  - Transdimensional MCMC (out of scope)
- If you care about parameter identifiability, check!